MATHEMATICAL MODELLING OF THE SOIL DESALINIZATION PROCESS

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Of the many physical and chemical factors influencing the migration of salt in the ground, only molecular-filtration diffusion, convective transport, and salt exchange between the porous soil skeleton and the moving solution are usually taken into account [1]. The salt-exchange process is of a diffusion nature ("internal" diffusion) and depends on the types of soil and the nature of their salinization. We may assume that soils of a heavy mechanical composition (clay, heavy loam) possessing high moisture-retention capacity are most appropriately modelled by a heterogeneous porous medium with porosity $m = m_1 + m_2$, where m_1 denotes the volume of transit pores occupied by the moving solution and m_2, the volume of terminal pores, filled with a stationary solution ("bound" moisture). In this case, the internal diffusion mass-exchange process between both types of pores is described [2, 3] in the form of Freundlich-type isotherms,

$$\alpha\,\partial N/\partial t = c - N, \tag{0.1}$$

where c (x, t) and N (x, t) are the concentrations of solutions in the transit and terminal pores, respectively, t is the time, x is a coordinate, and α is the kinetic parameter. Salinization of soil of light mechanical composition (sand, light loam) with low moisture-retention capacity is associated with the presence of salts in the solid phase. We must write kinetic equations for salt dissolution in place of Eq. (0.1). Some types of these equations have been previously presented [1-5]. In this work, three problems modelling the soil desalinization process and admitting analytic solutions for arbitrarily given initial salinization are considered. It is assumed here that the "internal" diffusion process occurs sufficiently rapidly (or infinitely rapidly) in comparison with the "external" diffusion process and convection.

1. Convective Salt Transport in a Heterogeneous Porous Medium

Ignoring the influence of the external diffusion process, we will write a balance equation for a salt mass in a moving solution:

$$vc_x + m_1 c_t + m_2 N_t = 0 \tag{1.1}$$

(v is the filtration rate of the solution through transit pores). We may assume without loss of generality that the concentration of wash water is zero, and the boundary conditions for the system of Eqs. (0.1), (1.1) applied to the washing problem for saline soil initially free of bulk moisture, are written in the form

$$c(0, t) = 0;\ N(x, m_1 x/v) = \varphi(x). \tag{1.2}$$

The solution of this problem is found using the Riemann method [6] and has the form

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$$c(x,t) = e^{-x_2}\int_0^{x_1} I_0(2\sqrt{\xi x_2})\,\varphi_0(x_1-\xi)\,e^{-\xi}d\xi;$$

$$N(x,t) = c(x,t) + e^{-x_2}\left[I_0(2\sqrt{x_1x_2})\,\varphi_0(0)\,e^{-x_1} + \int_0^{x_1} I_0(2\sqrt{\xi x_2})\varphi_0'(x_1-\xi)\,e^{-\xi}d\xi\right],$$

$$x_1 = m_2x/(v\alpha);\quad x_2 = (t - m_1x/v)/\alpha;$$

$\varphi_0(x_1) = \varphi(v\alpha x_1/m_2$; $I_0(z)$ is a modified zero-order Bessel function of the first kind, and the prime denotes differentiation.

If the internal diffusion process occurs sufficiently rapidly, equations of the type of (0.1) are sometimes replaced by one of the following assumptions [7]:

$$N(x,\ t) = c(x+\varkappa,\ t); \tag{A}$$

$$N(x,\ t) = c(x,\ t-\varkappa_1), \tag{B}$$

where $\varkappa$ is the "lag path" and $\varkappa_1$ is the "time lag," both assumed to be small.

Assuming that the decompositions $c(x+\varkappa,\ t) \approx c(x,\ t) + \varkappa c_x$ and $c(x,\ t-\varkappa_1) \approx c(x,\ t) - \varkappa_1 c_1$ are valid and substituting them in Eq. (1.1), we obtain an equation for the "effective plate" method [assumption (A)],

$$vc_x + mc_t + m_2\varkappa c_{xt} = 0 \tag{1.3}$$

or an equation for the "time lag" method [assumption (B)],

$$vc_x + mc_t - m_2\varkappa_1 c_{tt} = 0. \tag{1.4}$$

Equation (1.3) is hyperbolic, while Eq. (1.4) is parabolic. Consequently, the formulation of the boundary-value problems and the properties of the solutions of these equations will differ. Assumptions (A) and (B) are in some sense equivalent. This may be verified. Since the functions c and N depend on the parameter $\varkappa$ (or $\varkappa_1$), assumptions (A) and (B), correspondingly take the form

$$N(x,t,\varkappa) = c(x+\varkappa,t,\varkappa) = c_0(x,t) + \left[c_1(x,t) + \frac{\partial c_0}{\partial x}(x,t)\right]\varkappa + 0(\varkappa^2);$$

$$N(x,t,\varkappa_1) = c(x,t-\varkappa_1,\varkappa_1) = c_0(x,t) + \left[c_1(x,t) - \frac{\partial c_0}{\partial t}(x,t)\right]\varkappa_1 + 0(\varkappa_1^2),$$

where

$$c_1(x,\ t) = c_\varkappa(x,\ t,\ 0),\quad (c_1(x,t) = c_{\varkappa_1}(x,t,0)).$$

Substituting these decompositions in Eq. (1.1) and conditions (1.2), we obtain the problem

$$\begin{cases} v\dfrac{\partial c_0}{\partial x} + m\dfrac{\partial c_0}{\partial t} = 0, & 0 < x < \dfrac{vt}{m_1}, \quad t > 0, \\ c_0(0,t) = 0, & c_0(x, m_1x/v) = \varphi(x); \end{cases} \tag{1.5}$$

$$\begin{cases} v\dfrac{\partial c_1}{\partial x} + m\dfrac{\partial c_1}{\partial t} = -m_2\dfrac{\partial^2 c_0}{\partial x\partial t}\left(= m_2\dfrac{\partial^2 c_0}{\partial t^2}\right), \\ c_1(0,t) = 0, \quad c_1(x, m_1x/v) = -c_{0x}(= c_{0t}). \end{cases} \tag{1.6}$$

for the functions $c_0(x,\ t)$ and $c_1(x,\ t)$ by equating coefficients in powers of $\varkappa$ (or $\varkappa_1$).

Let us set $\tau = vt/m_1$, and $\lambda = m_1/m$, so that the solution of the problem (1.5) will be written in the form

$$c_0(x,t) = \begin{cases} \varphi(z) & \text{for } \lambda\tau \leqslant x \leqslant \tau, \\ 0 & \text{for } 0 \leqslant x < \lambda\tau; \end{cases}$$
$$z = (x = \lambda\tau)/(1-\lambda).$$

Assuming the $\varphi(x)$ is twice differentiable, the problem (1.6) reduces to the form

$$\frac{\partial c_1}{\partial x}+\frac{1}{\lambda}\frac{\partial c_1}{\partial t}=\frac{1}{1-\lambda}\varphi''(z)\left(=\frac{v}{m_2}\varphi''(z)\right),$$
$$c_1(0,t)=0;\ c_1(x,x)=-\varphi'(x)/(1-\lambda)\ (=-v\varphi'(x)/m_2). \tag{1.7}$$

It is therefore evident that hypotheses (A) and (B) are equivalent if $\varkappa = v\varkappa_1/m$.

The function

$$c_1(x,t)=\begin{cases}-(1-\lambda)^{-1}\varphi'(z)-\lambda(1-\lambda)^{-2}(x-\tau)\varphi''(z) & \text{for } \lambda\tau\leqslant x\leqslant\tau,\\ 0 & \text{for } 0\leqslant x<\lambda\tau.\end{cases} \tag{1.8}$$

will be the solution of the problem (1.7).

We may arrive at the same result by finding the solution of Eqs. (0.1), (1.1) under the conditions (1.2) in the form of series in powers of the parameter α and retaining only two terms of the decompositions. It turns out that $\varkappa_1 = \alpha$ and, in view of Eq. (1.8), this solution to within magnitudes on the order of α inclusively, will be written in the form

$$c(x,t,\alpha)=\begin{cases}\varphi(z)-\frac{v\alpha}{m_2}\left[\varphi'(z)-\frac{m_1}{m_2}\left(x-\frac{vt}{m_1}\right)\varphi''(z)\right], & \frac{vt}{m}\leqslant x\leqslant\frac{vt}{m_1},\\ 0, & 0\leqslant x\leqslant vt/m;\end{cases}$$
$$N(x,t,\alpha)=\begin{cases}\varphi(z)-\frac{v\alpha m_1}{m_2^2}\left(x-\frac{vt}{m_1}\right)\varphi''(z), & \frac{vt}{m}\leqslant x\leqslant\frac{vt}{m_1},\\ 0, & 0\leqslant x\leqslant vt/m.\end{cases}$$

2. Convective Diffusion in a Heterogeneous Porous Medium

First Approximation. The influence of the external diffusion process reduces to the addition of a term Dc_{xx} to the right side of Eq. (1.1), where D is the diffusion coefficient. The first approximation $c_0(x, t) \equiv N_0(x, t)$ and the first terms of the decompositions of $c(x, t, \alpha)$ and $N(x, t, \alpha)$ in powers of α will be determined in the case of comparatively rapid salt exchange between the transit and terminal pores (small α) by the conditions

$$\begin{cases}D\frac{\partial^2 c_0}{\partial x^2}-v\frac{\partial c_0}{\partial x}=m\frac{\partial c_0}{\partial t}; & 0<x<x_0(t),\quad t>0,\\ c_0=c_n,\quad x=0; & c_0=N_0=\varphi_0(x),\quad t=m_1x/v,\end{cases} \tag{2.1}$$

where c_n is the concentration of irrigation water, $\varphi_0(x)$ is the initial salt concentration in the solution "bound" to the soil, and $x_0 = vt/m_1$ is the forward front of irrigation water.

We set $u=c_0-c_n$, $\xi=vx/D$, $\tau=v^2t/(m_1D)$, and $\varphi(\xi)=\varphi_0(D\xi/v)-c_n$, so that the problem (2.1) takes the form

$$u_{\xi\xi}-u_\xi=\lambda^{-1}u_\tau,\quad 0<\xi<\tau,\ \tau>0; \tag{2.2}$$

$$\xi=0,\ u=0,\ \xi=\tau,\ u=\varphi(\tau). \tag{2.3}$$

Suppose $u(\xi, \tau)$ is continued as a solution of Eq. (2.2) in the entire quadrant $\xi > 0$, $\tau > 0$ and let $\rho(\xi) = u(\xi, 0)$. The substitution for the desired function $u = w\exp(\xi/2-\lambda\tau/4)$ leads to the problem

$$w_{\xi\xi}=\lambda^{-1}w_\tau,\ \xi>0,\ \tau>0,$$
$$\xi=0,\ w=0,\ \tau=0,\ w=\rho_1(\xi)=\rho(\xi)\exp(-\xi/2),$$

whose solution will be given by the function [8]

$$w = \frac{1}{2\sqrt{\pi\lambda\tau}} \int_0^\infty \rho_1(s) \left\{\exp\left[-\frac{(\xi - s)^2}{4\lambda\tau}\right] - \exp\left[-\frac{(\xi + s)^2}{4\lambda\tau}\right]\right\} ds.$$

To determine the function $\rho_1(\xi)$ by means of the second condition (2.3), we obtain the integral equation

$$\frac{1}{\sqrt{\pi\lambda\tau}} \int_0^\infty F(s, \lambda) \exp\left(-\frac{s^2}{4\lambda\tau}\right) ds = e^{\gamma\tau}\varphi(\tau), \tag{2.4}$$

$$(F(s, \lambda) = \rho_1(s) \operatorname{sh} [s/(2\lambda)], \ \gamma = (1 - \lambda)^2/(4\lambda)).$$

We let

$$L_p(\varphi) = \int_0^\infty \varphi(s) e^{-ps} ds; \quad L_s^{-1}(L_p(\varphi)) \equiv \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} L_p(\varphi) e^{ps} dp = \varphi(s)$$

denote the direct and inverse Laplace transformation of the function $\varphi(s)$. Applying L_p to both sides of Eq. (2.4), we find [9]

$$\sqrt{\lambda p}\, L_{p-\gamma}(\varphi) = \int_0^\infty F(s, \lambda) e^{-s\sqrt{p/\lambda}} ds.$$

Again replacing $\sqrt{p/\lambda}$ by p, we arrive at the equation

$$\lambda p L_{\lambda p^2 - \gamma}(\varphi) = L_p(F)$$

and, now applying the operation L_s^{-1}, we obtain the equation

$$\rho_1(s) = \operatorname{sh}^{-1}[s/(2\lambda)]\, L_s^{-1}[\lambda p L_{\lambda p^2 - \gamma}(\varphi)],$$

which completes the solution of the initial problem (2.1).

Example. Suppose $\varphi(\tau) = \exp(\mu\tau)$. Then

$$L_p(\varphi) = (p - \mu)^{-1}; \quad \lambda p L_{\lambda p^2 - \gamma}(\varphi) = p[p^2 - (\mu + \gamma)/\lambda]^{-1}.$$

After calculating the residues at the points $p = \pm\sqrt{(\mu + \gamma)/\lambda}$, we obtain the result

$$\rho_1(s) = \operatorname{ch}[s\sqrt{(\mu + \gamma/\lambda)}]\, \operatorname{sh}^{-1}[s/(2\lambda)].$$

When $\mu = 0$, we obtain the case of depth-constant salinization,

$$\rho_1(s) = \operatorname{ch}[s(1 - \lambda)/(2\lambda)]\, \operatorname{sh}^{-1}[s/(2\lambda)].$$

The solution of (2.1) or (2.2), (2.3) is unique. This will be proved as was done in [10]. We find, by considering the function

$$H(\tau) = \int_0^\tau u^2(\xi, \tau)\, d\xi = 0,$$

where $u = u_1 - u_2$ is the difference between the two solutions of Eq. (2.2) satisfying the homogeneous conditions $u(0, \tau) = u(\tau, \tau) = 0$, that

$$H'(\tau) = -2\lambda \int_0^\tau u_\xi^2(\xi, \tau)\, d\xi \leqslant 0.$$

Since $H(0) = 0$, $H(\tau) \equiv 0$, and, consequently, the homogeneous problem has only a trivial solution.

3. Convective diffusion in a Porous Medium with Instantaneous Salt Dissolution

Let us consider the case when soil of light mechanical composition contains highly soluble salts in the solid phase. We assume that the salt instantaneously dissolves at the moment the drench boundary $x = x_0(t)$ is reached, and subsequently moves forward due to the action of convective transfer and filtration diffusion. The problem reduces to finding the function $c(x, t)$ by means of the condition

$$\begin{cases} Dc_{xx} - vc_x = mc_t, & 0 < x < x_0(t), \quad t > 0, \\ c = c_n, \quad x = 0, & Dc_x = N(x)\dot{x}_0, \quad x = x_0 = vt/m, \end{cases} \tag{3.1}$$

where $N(x)$ is the initial density of distribution of salts in physical space (one particular case of this problem, $N(x) = N_0 = \text{const}$, has been previously considered [10]). We rewrite the problem (3.1) in dimensionless variables ξ, τ:

$$\begin{cases} u_{\xi\xi} - u_\xi = u_\tau, \ 0 < \xi < \tau, \ \tau > 0, \\ u = 0, \ \xi = 0, \ u_\xi = \varphi_1(\tau), \ \xi = \tau \end{cases}$$
$$(u = c - c_n, \ \xi = vx/D, \ \tau = v^2t/(mD), \quad \varphi_1(\tau) = N(D\tau/v)/m).$$

As in the solution of the problem (2.2), (2.3), we represent the function $u(\xi, \tau)$ in the form

$$u(\xi, \tau) = \frac{\exp(\xi/2 - \tau/4)}{2\sqrt{\pi\tau}} \int_0^\infty \rho_1(s)\left\{\exp\left[-\frac{(\xi - s)^2}{4\tau}\right] - \exp\left[-\frac{(\xi + s)^2}{4\tau}\right]\right\} ds.$$

The desired function $\rho_1(x)$ with initial data

$$\rho_1(0) = 0 \tag{3.2}$$

will be found from the boundary condition at $\xi = \tau$. We may verify, using simple computations, that the latter leads to the integral equation

$$\frac{1}{\sqrt{\pi\tau}} \int_0^\infty F_1(s) \exp\left(-\frac{s^2}{4\tau}\right) ds = \varphi_1(\tau)$$
$$(F_1(s) = [\rho_1(s)\,\mathrm{ch}\,(s/2)]'),$$

which coincides with Eq. (2.4) when $\lambda = 1$ ($\gamma = 0$). Thus

$$F_1(s) = F(s, 1) = L_s^{-1}[pL_{p^2}(\varphi_1)].$$

Integrating $F_1(s)$ over s and taking into account condition (3.2), we obtain

$$\rho_1(s) = \mathrm{sh}^{-1}(s/2) \int_0^s L_s^{-1}[pL_{p^2}(\varphi_1)]\, ds.$$

Example. Suppose $\varphi_1 = \exp(\mu\tau)$, so that $L_p(\varphi_1) = (p - \mu)^{-1}$. When $\mu > 0$ (salinization increases with depth),

$$\rho_1(s) = \mathrm{sh}\,(s\sqrt{\mu})\, \mu^{-1/2}\, \mathrm{ch}^{-1}(s/2);$$

when $\mu < 0$ (salinization decreases with depth),

$$\rho_1(s) = \sin\,(s\sqrt{-\mu}(-\mu)^{-1/2}\, \mathrm{ch}^{-1}(s/2);$$

when $\mu = 0$ (uniform salinization),

$$\rho_1(s) = s\ \mathrm{ch}^{-1}(s/2).$$

The last equation coincides with a previously obtained result [10].

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